

Introduction to Homological Algebra

Part 4 Examples and applications

§4.1. Theorem (Grothendieck): If k is any field, then $K_0(\text{Coh}(\mathbb{P}_k^n))$ is the free abelian group on the classes

↑ category of coherent sheaves

$$[\mathcal{O}], [\mathcal{O}(-1)], \dots, [\mathcal{O}(-n)]$$

§4.2. How can we describe the category $D^b(\text{Coh}(\mathbb{P}_k^n))$ itself?

Fact: $D^b(\mathbb{P}_k^n)$ is generated, as a triangulated category, by the objects $\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2), \dots, \mathcal{O}(-n)$.

Let us write $V = k^{n+1}$, so that

$$S = \text{Sym}(V)$$

$$\mathbb{P}_k^n = (V \setminus \{0\}) / \mathbb{G}_m, k$$

Let S be the symmetric algebra of V . Let \mathcal{A} be the category of free S -modules with generators in degrees $[0, n]$.

Theorem. There is an equivalence of triangulated categories

$$K^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathbb{P}_k^n)$$

so that $S(a) \longmapsto \mathcal{O}(-a)$

(The proof is not very difficult.)

§4.3. Two Fourier transforms

First: A special case of Koszul duality, found by Bernstein - Gelfand - Gelfand.

Let V be a finite dimensional vector space over a field k and write

$$S = \text{Sym}(V), \quad \Lambda = \Lambda(V^*).$$

Write $\mathcal{D}_S =$ the derived category of the abelian category of graded S -modules,

and similarly for \mathcal{D}_Λ .

There is a natural equivalence of triangulated categories $\mathcal{D}_S \xrightarrow{\sim} \mathcal{D}_\Lambda$, called the Fourier transform.

Note that the abelian categories of (graded) S -modules and Λ -modules are very different!

How is the Fourier transform defined?

Note that S -grmod and Λ -grmod have natural monoidal structures, given by the tensor product \otimes_S and convolution $*_\Lambda$, respectively. The convolution $*_\Lambda$ comes from the standard coalgebra structure on Λ .

Definition of the Fourier transform.

Consider

$$c \in V \otimes_k V^* \subset S \otimes_k \Lambda$$

↑
standard Casimir element

One checks that $c^2 = 0$, hence multiplication by c on the right defines a differential on $S \otimes_k \Lambda$. We get what is known as the standard Koszul complex, call it K .

The Fourier transform functors

$$\mathcal{D}_S \rightleftarrows \mathcal{D}_\Lambda$$

are $- \otimes_S K$ and $K \otimes_\Lambda -$.

Note that we need to use two different gradings on K , one coming from S and another coming from Λ .

§4.4.

Fourier-Mukai transform

A = an abelian variety

A° = the dual abelian variety

The Fourier-Mukai transform contains many of the classical results about the cohomology of coherent sheaves on A .

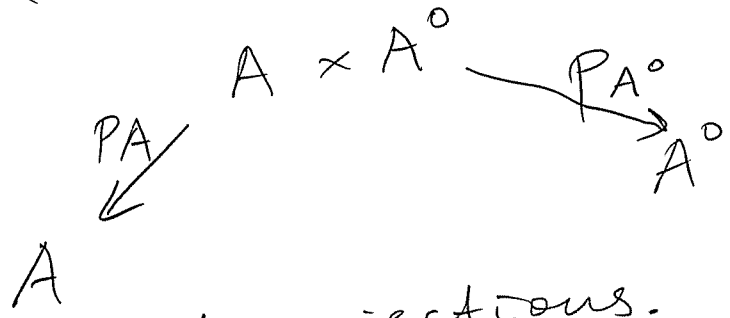
Consider the universal line bundle \mathcal{L} on $A \times A^\circ$ (called the Poincaré line bundle). We get a functor

$$\text{Four}_{\mathcal{L}} : D^b(A) \longrightarrow D^b(A^\circ)$$

defined by

$$\text{Four}_{\mathcal{L}}(M) = R(p_{A^\circ})_* (\mathcal{L} \otimes p_A^* M),$$

where



are the natural projections.

Theorem: $\text{Four}_{\mathcal{L}}$ is an equivalence of triangulated categories

A quasi-inverse is given by a similar formula.

§4.5. Remarks

The Fourier-Mukai transform interchanges the tensor product and convolution on D^b . It also interchanges delta-sheaves and topologically trivial line bundles.

Suggestion: look through Verdier's thesis