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Introduction to Homological Algebra

Part 4 Examples and applications

§4.1. **Theorem** (Grothendieck): If k is any field, then $K_0(\mathrm{Coh}(\mathbb{P}^n_k))$ is the free abelian group on the classes

category of coherent sheaves

$$[\mathcal{O}], \ [\mathcal{O}(-1)], \dots, [\mathcal{O}(-n)]$$

§4.2. How can we describe the category $D^b(\mathbb{P}^n_k) \underset{=}{\text{def}} D^b(\mathrm{Coh}(\mathbb{P}^n_k))$ itself?

Fact: $D^b(\mathbb{P}^n_k)$ is generated, as a triangulated category, by the objects $\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2), \dots, \mathcal{O}(-n)$.

Let us write $V = k^{n+1}$, so that $\mathbb{P}^n_k = (V \setminus \{0\}) / \mathbb{G}_{m,k}$. Let $S = \mathrm{Sym}(V)$ be the symmetric algebra of V. Let $\mathcal{A}$ be the category of free S-modules with generators in degrees $[0, n]$.

<u>Theorem</u>. There is an equivalence of triangulated categories

$$K^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathbb{P}^n_k)$$

so that $S(a) \longmapsto \mathcal{O}(-a)$

(The proof is not very difficult.)

§4.3. Two Fourier transforms

<u>First</u>: A special case of Koszul duality, found by Bernstein - Gelfand - Gelfand.

Let V be a finite dimensional vector space over a field k and write

$$S = \mathrm{Sym}(V), \qquad \Lambda = \Lambda(V^*).$$

Write $\mathcal{D}_S$ = the derived category of the abelian category of graded S-modules,

and similarly for $\mathcal{D}_\Lambda$.

There is a natural equivalence of triangulated categories $\mathcal{D}_S \xrightarrow{\sim} \mathcal{D}_\Lambda$, called the Fourier transform.

Note that the abelian categories of (graded) S-modules and Λ-modules are very different!

How is the Fourier transform defined? Note that S-grmod and Λ-grmod have natural monoidal structures, given by the tensor product $\otimes_S$ and convolution $*_\Lambda$, respectively. The convolution $*_\Lambda$ comes from the standard coalgebra structure on Λ.

<u>Definition</u> <u>of the</u> <u>Fourier transform</u>.

Consider

$$c \in V \otimes_k V^* \subset S \otimes_k \Lambda$$

$$\uparrow$$

standard Casimir element

One checks that $c^2 = 0$, hence multiplication by c on the right defines a differential on $S \otimes_k \Lambda$. We get what is known as the standard <u>Koszul</u> <u>complex</u>, call it K.

The Fourier transform functors

$$\begin{tikzcd} \mathcal{D}_S \arrow[r, bend left] & \mathcal{D}_\Lambda \arrow[l] \end{tikzcd}$$

are $- \otimes_S K$ and $K \otimes_\Lambda -$.

Note that we need to use two different gradings on K, one coming from S and another coming from Λ.

§4.4. Fourier–Mukai transform

A = an abelian variety

A° = the dual abelian variety

The Fourier–Mukai transform contains many of the classical results about the cohomology of coherent sheaves on A.

Consider the universal line bundle $\mathcal{L}$ on $A \times A^\circ$ (called the Poincaré line bundle). We get a functor

$$\mathrm{Four}_{\mathcal{L}} : D^b(A) \longrightarrow D^b(A^\circ)$$

defined by

$$\mathrm{Four}_{\mathcal{L}}(M) = R(p_{A^\circ})_*\left(\mathcal{L} \otimes p_A^* M\right),$$

where

$$\begin{tikzcd} & A \times A^\circ \arrow[dl, "p_A"'] \arrow[dr, "p_{A^\circ}"] & \\ A & & A^\circ \end{tikzcd}$$

are the natural projections.

<u>**Theorem**</u>: $\mathrm{Four}_{\mathcal{L}}$ is an equivalence of triangulated categories

A quasi-inverse is given by a similar formula.

§4.5. Remarks

The Fourier–Mukai transform interchanges the tensor product and convolution on D^b. It also interchanges delta-sheaves and topologically trivial line bundles.

Suggestion: look through Verdier's thesis